Elementary Mathematics II (92)

Unit V

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5.1 Summation of Trigonometric Series

(1) Sum of Sines of n angles when angles are in Arithmetic Progression

Let

$\alpha, \alpha+\beta, \alpha+2\beta, \ldots, \alpha+(n-1)\beta$ are in AP

then, the Sine ratios of these angles are

$\sin\alpha, \sin(\alpha+\beta), \sin(\alpha+2\beta), \ldots, \sin(\alpha+(n-1)\beta)$

their sum is

$$S_n = \sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \cdots + \sin[\alpha+(n-1)\beta]$$

Multiplying both sides by $2\sin\left(\frac{\beta}{2}\right)$

$$2\sin\left(\frac{\beta}{2}\right) S_n = 2\sin\alpha \cdot \sin\left(\frac{\beta}{2}\right) + 2\sin(\alpha+\beta)\sin(\beta/2) + 2\sin(\alpha+2\beta)\sin(\beta/2) + \cdots$$
$$+ 2\sin(\alpha+(n-1)\beta)\sin(\beta/2) \quad \longrightarrow (1)$$

Considering the terms in the RHS of (1),

First term, $2\sin\alpha \sin\left(\frac{\beta}{2}\right) = \cos\left(\alpha-\frac{\beta}{2}\right) - \cos\left(\alpha+\frac{\beta}{2}\right)$ $\left\{\because 2\sin A \sin B = \cos(A-B) - \cos(A+B)\right.$

Second term, $2\sin(\alpha+\beta)\sin\left(\frac{\beta}{2}\right) = \cos\left(\alpha+\beta-\frac{\beta}{2}\right) - \cos\left(\alpha+\beta+\frac{\beta}{2}\right)$

$$= \cos\left(\alpha+\frac{\beta}{2}\right) - \cos\left(\alpha+\frac{3\beta}{2}\right)$$

3rd term, $2\sin(\alpha+2\beta)\cdot\sin\left(\frac{\beta}{2}\right) = \cos\left(\alpha+2\beta-\frac{\beta}{2}\right) - \cos\left(\alpha+2\beta+\frac{\beta}{2}\right)$

$$= \cos\left(\alpha+\frac{3\beta}{2}\right) - \cos\left(\alpha+\frac{5\beta}{2}\right)$$

$\ldots \quad \ldots \quad \ldots$

n^{th} term, $2\sin[\alpha+(n-1)\beta]\sin\left(\frac{\beta}{2}\right) = \cos\left[\alpha+\left(n-\frac{3}{2}\right)\beta\right] - \cos\left[\alpha+\left(n-\frac{1}{2}\right)\beta\right]$

Adding all the terms, we get

$$2\sin\left(\frac{\beta}{2}\right) S_n = \cos\left(\alpha-\frac{\beta}{2}\right) - \cos\left[\alpha+\left(n-\frac{1}{2}\right)\beta\right]$$

$\boxed{\cos C - \cos D = 2\sin\frac{C+D}{2}\sin\frac{D-C}{2}}$

$$= 2\sin\left[\frac{\alpha-\frac{\beta}{2}+\alpha+\left(n-\frac{1}{2}\right)\beta}{2}\right]\sin\left[\frac{\alpha+\left(n-\frac{1}{2}\right)\beta-(\alpha-\beta/2)}{2}\right]$$

$$= 2\sin\left(\frac{2\alpha + \left(-\frac{1}{2}+n-\frac{1}{2}\right)\beta}{2}\right)\sin\left(\frac{n\beta}{2}\right)$$

$$= 2\sin\left(\frac{2\alpha + (n-1)\beta}{2}\right)\sin\left(\frac{n\beta}{2}\right)$$

$$2\sin\frac{\beta}{2}\, S_n = 2\sin\left(\alpha + \frac{(n-1)\beta}{2}\right)\sin\left(\frac{n\beta}{2}\right)$$

$$\therefore \boxed{S_n = \frac{\sin\left(\alpha + \left(\frac{n-1}{2}\right)\beta\right)\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}}$$

(2) Sum of Cosines of n angles when angles are in A.P.

Let $\alpha, \alpha+\beta, \alpha+2\beta, \ldots, \alpha+(n-1)\beta$ be n angles in A.P.

Sum of their Cosine ratio's

$$S_n = \cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \cdots + \cos(\alpha+(n-1)\beta)$$

Multiplying both Sides by $2\sin\left(\frac{\beta}{2}\right)$

$$2\sin\left(\frac{\beta}{2}\right) S_n = 2\cos\alpha\sin\left(\frac{\beta}{2}\right) + 2\cos(\alpha+\beta)\sin\left(\frac{\beta}{2}\right) + 2\cos(\alpha+2\beta)\sin\left(\frac{\beta}{2}\right) + \cdots + 2\cos(\alpha+(n-1)\beta)\sin\left(\frac{\beta}{2}\right) \quad \to (1)$$

Considering the terms in the RHS of (1), we get

First term, $2\cos\alpha\sin\left(\frac{\beta}{2}\right) = \sin\left(\alpha+\frac{\beta}{2}\right) - \sin\left(\alpha-\frac{\beta}{2}\right)$

Second term, $2\cos(\alpha+\beta)\sin\left(\frac{\beta}{2}\right) = \sin\left(\alpha+\frac{3\beta}{2}\right) - \sin\left(\alpha+\frac{\beta}{2}\right)$

3rd term, $2\cos(\alpha+2\beta)\sin\left(\frac{\beta}{2}\right) = \sin\left(\alpha+\frac{5\beta}{2}\right) - \sin\left(\alpha+\frac{3\beta}{2}\right)$

... ...

n^{th} term, $2\cos[(\alpha+(n-1)\beta)]\sin\left(\frac{\beta}{2}\right) = \sin\left(\alpha+\left(n-\frac{1}{2}\right)\beta\right) - \sin\left[\alpha+\left(n-\frac{3}{2}\right)\beta\right]$

Adding all the terms, we get

$$2\sin\left(\frac{\beta}{2}\right)\cdot S_n = \sin\left[\alpha+\left(n-\frac{1}{2}\right)\beta\right] - \sin\left(\alpha-\frac{\beta}{2}\right)$$

$$= 2\cos\left[\frac{\alpha+\left(n-\frac{1}{2}\right)\beta + \left(\alpha-\frac{\beta}{2}\right)}{2}\right]\sin\left[\frac{\alpha+\left(n-\frac{1}{2}\right)\beta - \left(\alpha-\frac{\beta}{2}\right)}{2}\right]$$

$$2\sin\left(\frac{\beta}{2}\right) S_n = 2\cos\left[\alpha + \left(\frac{n-1}{2}\right)\beta\right] \sin\left(\frac{n\beta}{2}\right)$$

$$\boxed{\therefore S_n = \frac{\cos\left[\alpha + \left(\frac{n-1}{2}\right)\beta\right] \sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}}$$

5.2 Sum of the hyperbolic trigonometric Series

① $\sinh \alpha + \sinh(\alpha+\beta) + \sinh(\alpha+2\beta) + \dots$ upto n terms

Solution:

WKT ~~Sin~~

$$\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \dots = \frac{\sin\left[\alpha + \left(\frac{n-1}{2}\right)\beta\right] \sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$$

Replacing α by $i\alpha$ and β by $i\beta$ respectively

$$\sin(i\alpha) + \sin i(\alpha+\beta) + \sin i(\alpha+2\beta) + \dots = \frac{\sin i\left[\alpha + \left(\frac{n-1}{2}\right)\beta\right] \sin\left(i\frac{n\beta}{2}\right)}{\sin\left(\frac{i\beta}{2}\right)}$$

$$i\sinh\alpha + i\sinh(\alpha+\beta) + i\sinh(\alpha+2\beta) + \dots = \frac{i\sinh\left[\alpha + \left(\frac{n-1}{2}\right)\beta\right] i\sinh\left(\frac{n\beta}{2}\right)}{i\sinh\left(\frac{\beta}{2}\right)}$$

~~i [Sin~~

$$i\left\{\sinh\alpha + \sinh(\alpha+\beta) + \sinh(\alpha+2\beta) + \dots\right\} = \frac{i\sinh\left(\alpha + \left(\frac{n-1}{2}\right)\beta\right) i\sinh\left(\frac{n\beta}{2}\right)}{i\sinh\left(\frac{\beta}{2}\right)}$$

$$\Rightarrow \boxed{\sinh\alpha + \sinh(\alpha+\beta) + \sinh(\alpha+2\beta) + \dots = \frac{\sinh\left[\alpha + \left(\frac{n-1}{2}\right)\beta\right] \sinh\left(\frac{n\beta}{2}\right)}{\sinh\left(\frac{\beta}{2}\right)}}$$

$$② \boxed{\cosh\alpha + \cosh(\alpha+\beta) + \cosh(\alpha+2\beta) + \dots = \frac{\cosh\left[\alpha + \left(\frac{n-1}{2}\right)\beta\right] \sinh\left(\frac{n\beta}{2}\right)}{\sinh\left(\frac{\beta}{2}\right)}}$$

Example Find the sum of the following trigonometric series

(i) $\sin\alpha + \sin 2\alpha + \sin 3\alpha + \dots$ upto n terms

(ii) $\sin\alpha + \sin 3\alpha + \sin 5\alpha + \dots$ upto n terms

(iii) $\cos\alpha + \cos 2\alpha + \cos 3\alpha + \dots$ upto n terms

(iv) $\cos\alpha + \cos 3\alpha + \cos 5\alpha + \dots$ upto n terms

Solution

(i) WKT $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \dots$ upto n terms

Sum $S_n = \dfrac{\sin\left[\alpha + \left(\frac{n-1}{2}\right)\beta\right] \sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)} \longrightarrow (1)$

where α - First term, β - Common difference

$S_n = \sin\alpha + \sin 2\alpha + \sin 3\alpha + \dots$ upto n terms

First term $\alpha = \alpha$

Common diff. $\beta = 2\alpha - \alpha$

$\beta = \alpha$

$$= \frac{\sin\left[\alpha + \left(\frac{n-1}{2}\right)\alpha\right] \sin\left(\frac{n\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)}$$

$$= \frac{\sin\left(\frac{2\alpha + n\alpha - \alpha}{2}\right) \sin\left(\frac{n\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)}$$

$$= \frac{\sin\left(\frac{\alpha + n\alpha}{2}\right) \sin\left(\frac{n\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)}$$

$$\boxed{S_n = \frac{\sin\left(\left(\frac{n+1}{2}\right)\alpha\right) \sin\left(\frac{n\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)}}$$

(ii) $S_n = \sin\alpha + \sin 3\alpha + \sin 5\alpha + \dots$ upto n terms

First term $\alpha = \alpha$

C.d $\beta = 3\alpha - \alpha$

$\beta = 2\alpha$

$$= \frac{\sin\left[\alpha + \left(\frac{n-1}{2}\right)2\alpha\right] \sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{2\alpha}{2}\right)} \quad \{\because \text{ from } (1)$$

$$= \frac{\sin(n\alpha)\, \sin\left(\frac{n2\alpha}{2}\right)}{\sin(\alpha)}$$

$$\boxed{S_n = \frac{\sin^2 n\alpha}{\sin\alpha}}$$

(iii) $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \cos(\alpha+3\beta) + \dots$ upto n terms.

~~St~~ Their Sum $S_n = \dfrac{\cos\left[\alpha + \left(\frac{n-1}{2}\right)\beta\right]\sin\left(\frac{n\beta}{2}\right)}{\sin(\beta/2)} \rightarrow$ (2)

$\cos\alpha + \cos 2\alpha + \cos 3\alpha + \dots$ upto n terms

First term $\alpha = \alpha$

C.D $\beta = 2\alpha - \alpha$

$\beta = \alpha$

$$S_n = \frac{\cos\left[\alpha + \left(\frac{n-1}{2}\right)\alpha\right]\sin\left(\frac{n\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)}$$

$$= \frac{\cos\left[\frac{2\alpha + (n-1)\alpha}{2}\right]\sin\left(\frac{n\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)}$$

$$= \frac{\cos\left(\frac{\alpha + n\alpha}{2}\right)\sin\left(\frac{n\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)}$$

$$\boxed{S_n = \frac{\cos\left(\frac{n+1}{2}\right)\alpha \ \sin\left(\frac{n\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)}}$$

(iv) $\cos\alpha + \cos 3\alpha + \cos 5\alpha + \dots$ n terms $= \dfrac{\cos\left[\alpha + \left(\frac{n-1}{2}\right)\beta\right]\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$

Here

First term $\alpha = \alpha$

Common difference $\beta = 3\alpha - \alpha$

$\beta = 2\alpha$

$$S_n = \frac{\cos\left[\alpha + \left(\frac{n-1}{2}\right)2\alpha\right]\cdot\sin\left(\frac{n\,2\alpha}{2}\right)}{\sin\left(\frac{2\alpha}{2}\right)}$$

$$= \frac{\cos\left[\alpha + (n-1)\alpha\right]\cdot\sin(n\alpha)}{\sin(\alpha)}$$

$2\sin A\cos A = \sin 2A$

$$= \frac{\cos(n\alpha)\ \sin(n\alpha)}{(\sin\alpha)}$$

$$\boxed{S_n = \frac{\sin 2(n\alpha)}{2\sin\alpha}}$$

Example Sum to n terms of the Series

$$\cos\frac{\pi}{2n+1} + \cos\frac{3\pi}{2n+1} + \cos\frac{5\pi}{2n+1} + \cdots$$

Solution

Given angles are,

$\frac{\pi}{2n+1}, \frac{3\pi}{2n+1}, \frac{5\pi}{2n+1} \cdots$ and they form a A.P

Here the first term $\alpha = \frac{\pi}{2n+1}$

and Common difference $\beta = \frac{3\pi}{2n+1} - \frac{\pi}{2n+1}$

$$\beta = \frac{2\pi}{2n+1}$$

$$\boxed{\text{WKT} \quad \cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \cdots n \text{ terms} = \frac{\cos\left[\alpha + \left(\frac{n-1}{2}\right)\beta\right] \sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}}$$

$S_n = \cos\frac{\pi}{2n+1} + \cos\frac{3\pi}{2n+1} + \cos\frac{5\pi}{2n+1} + \cdots n$ terms

$$= \frac{\cos\left[\frac{\pi}{2n+1} + \left(\frac{n-1}{2}\right)\frac{2\pi}{2n+1}\right] \sin\left(\frac{n}{2}\frac{2\pi}{2n+1}\right)}{\sin\left(\frac{1}{2}\frac{2\pi}{2n+1}\right)}$$

$$= \frac{\cos\left(\frac{\pi + n\pi - \pi}{2n+1}\right) \sin\left(\frac{n\pi}{2n+1}\right)}{\sin\left(\frac{\pi}{2n+1}\right)}$$

$$= \frac{\cos\left(\frac{n\pi}{2n+1}\right) \sin\left(\frac{n\pi}{2n+1}\right)}{\sin\left(\frac{\pi}{2n+1}\right)}$$

$$S_n = \frac{\sin 2\left(\frac{n\pi}{2n+1}\right)}{2\sin\left(\frac{\pi}{2n+1}\right)}$$

Example Prove that $\cos\frac{\pi}{11} + \cos\frac{3\pi}{11} + \cos\frac{5\pi}{11} + \cos\frac{7\pi}{11} + \cos\frac{9\pi}{11} = \frac{1}{2}$

Proof

$\frac{\pi}{11}, \frac{3\pi}{11}, \frac{5\pi}{11}, \frac{7\pi}{11}, \frac{9\pi}{11}$ are in AP

It has $n=5$ terms

First term $\alpha = \frac{\pi}{11}$

Common difference $\beta = \frac{2\pi}{11}$

$$S_n = \frac{\cos\left[\alpha + \left(\frac{n-1}{2}\right)\beta\right] \sin\left(\frac{n\beta}{2}\right)}{\sin(\beta/2)}$$

$$= \frac{\cos\left[\frac{\pi}{11} + \left(\frac{5-1}{2}\right)\frac{2\pi}{11}\right] \sin\left(\frac{5}{2}\right)\left(\frac{2\pi}{11}\right)}{\sin\frac{1}{2}\left(\frac{2\pi}{11}\right)}$$

$$= \frac{\cos\left[\frac{\pi}{11} + \frac{4\pi}{11}\right] \sin\left(\frac{5\pi}{11}\right)}{\sin\left(\frac{\pi}{11}\right)}$$

$$= \frac{\cos\left(\frac{5\pi}{11}\right) \sin\left(\frac{5\pi}{11}\right)}{\sin\left(\frac{\pi}{11}\right)}$$

$$= \frac{2 \sin\left(\frac{5\pi}{11}\right) \cos\left(\frac{5\pi}{11}\right)}{2 \sin\left(\frac{\pi}{11}\right)}$$

$\{\because 2\sin A\cos A = \sin 2A$

$$= \frac{\sin 2\left(\frac{5\pi}{11}\right)}{2 \sin\left(\frac{\pi}{11}\right)}$$

$$= \frac{\sin\left(\frac{10\pi}{11}\right)}{2 \sin\left(\frac{\pi}{11}\right)}$$

$$= \frac{\sin\left(\pi - \frac{\pi}{11}\right)}{2 \sin\left(\frac{\pi}{11}\right)}$$

$$= \frac{\sin\left(\frac{\pi}{11}\right)}{2 \sin\left(\frac{\pi}{11}\right)}$$

$$\boxed{S_n = \frac{1}{2}}$$

Example Find the sum of the series

$\sin\alpha\cos\alpha + \sin 2\alpha\cos 2\alpha + \sin 3\alpha\cos 3\alpha + \cdots$ up to n terms

Solution

$\boxed{\text{WKT} \quad \sin A\cos A = \frac{1}{2}\sin 2A}$

let

$S_n = \sin\alpha\cos\alpha + \sin 2\alpha\cos 2\alpha + \sin 3\alpha\cos 3\alpha + \cdots$ upto n terms

$= \frac{1}{2}\left[2\sin\alpha\cos\alpha + 2\sin 2\alpha\cos 2\alpha + 2\sin 3\alpha\cos 3\alpha + \cdots \text{ upto } n \text{ terms}\right]$

$= \frac{1}{2}\left[\sin 2\alpha + \sin 4\alpha + \sin 6\alpha + \cdots \text{ upto } n \text{ terms}\right]$

Here, $2\alpha, 4\alpha, 6\alpha \ldots$ are in AP

$= \frac{1}{2}\left[\frac{\sin\left[A + \left(\frac{n-1}{2}\right)B\right] \cdot \sin\left(\frac{nB}{2}\right)}{\sin\left(\frac{B}{2}\right)}\right]$

First term $A = 2\alpha$

Common differen $B = 2\alpha$

$= \frac{1}{2}\left[\frac{\sin\left[2\alpha + \left(\frac{n-1}{2}\right)2\alpha\right] \sin\left(\frac{n(2\alpha)}{2}\right)}{\sin\left(\frac{2\alpha}{2}\right)}\right]$

$= \frac{1}{2}\left[\frac{\sin\left[\alpha + n\alpha\right] \sin(n\alpha)}{\sin\alpha}\right]$

$= \frac{1}{2}\left[\frac{\sin(n+1)\alpha \cdot \sin n\alpha}{\sin\alpha}\right]$

Example If α be the exterior angle of a regular polygon of n sides, prove that

(i) $\sum_{r=0}^{n-1} \sin(\theta + r\alpha) = 0$ (ii) $\sum_{r=0}^{n-1} \cos(\theta + r\alpha) = 0$

Proof WKT the regular polygon has n sides

$\therefore$ The ~~no~~ number of exterior angles of regular polygon is n

Sum of all the exterior angles $= 2\pi$

Measure of Angle of each exterior angle, $\alpha = \frac{2\pi}{n}$.

$\sum_{r=0}^{n-1} \sin(\theta + r\alpha) = \sin\theta + \sin(\theta+\alpha) + \sin(\theta+2\alpha) + \cdots + \sin\left[\theta + (n-1)\alpha\right]$

In this series,

First term $a = \theta$, Common difference, $b = \alpha$

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$$= \frac{\sin\left[a+\left(\frac{n-1}{2}\right)b\right]\cdot\sin\left(\frac{nb}{2}\right)}{\sin\left(\frac{b}{2}\right)} \qquad \{\because \text{By formula}$$

$$= \frac{\sin\left[\theta+\left(\frac{n-1}{2}\right)\alpha\right]\ \sin\left(\frac{n\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)}$$

$$= \frac{\sin\left[\theta+\left(\frac{n-1}{2}\right)\frac{2\pi}{n}\right]\ \sin\left(\frac{n}{2}\cdot\frac{2\pi}{n}\right)}{\sin\left(\frac{2\pi}{2n}\right)} \qquad \{\because \alpha = \frac{2\pi}{n}$$

$$= \frac{\sin\left[\theta+(n-1)\frac{\pi}{n}\right]\ \sin\pi}{\sin\left(\frac{\pi}{n}\right)}$$

$$\{\because \sin\pi = 0$$

$$\sum_{r=0}^{n-1}\sin(\theta+r\alpha) = 0$$

(ii) $\sum_{r=0}^{n-1}\cos(\theta+r\alpha) = \cos(\theta) + \cos(\theta+\alpha) + \cos(\theta+2\alpha) + \cdots + \cos[(\theta+(n-1)\alpha]$

$$= \frac{\cos\left[\theta+\left(\frac{n-1}{2}\right)\alpha\right]\cdot\sin\left(\frac{n\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)}$$

$$\{\because \alpha = \frac{2\pi}{n}$$

$$= \frac{\cos\left[\theta+\left(\frac{n-1}{2}\right)\frac{2\pi}{n}\right]\cdot\sin\left(\frac{n}{2}\ \frac{2\pi}{n}\right)}{\sin\left(\frac{2\pi}{2n}\right)}$$

$$= \frac{\cos\left[\theta+\frac{(n-1)\pi}{n}\right]\ \sin\pi}{\sin\left(\frac{\pi}{n}\right)} \qquad \{\because \sin\pi = 0$$

$$\boxed{\sum_{r=0}^{n-1}\cos(\theta+r\alpha) = 0}$$

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Example: Sum to n terms the series

$$\sin^3\alpha + \sin^3 2\alpha + \sin^3 3\alpha + \cdots$$

Proof

W.K.T $\sin 3A = 3\sin A - 4\sin^3 A$

$$4\sin^3 A = 3\sin A - \sin 3A$$

$$\sin^3 A = \frac{1}{4}\left[3\sin A - \sin 3A\right]$$

Now, $T_1 = \sin^3\alpha = \frac{1}{4}(3\sin\alpha - \sin 3\alpha)$

$$T_2 = \sin^3 2\alpha = \frac{1}{4}(3\sin 2\alpha - \sin 6\alpha)$$

$$T_3 = \sin^3 3\alpha = \frac{1}{4}(3\sin 3\alpha - \sin 9\alpha)$$

$$T_4 = \sin^3 4\alpha = \frac{1}{4}(3\sin 4\alpha - \sin 12\alpha)$$

$$\cdots \quad \cdots \quad \cdots$$

$$T_n = \sin^3 n\alpha = \frac{1}{4}(3\sin n\alpha - \sin 3n\alpha)$$

Adding $T_1, T_2, T_3, T_4 \cdots T_n$ we get

~~$S_n = T_1 + T$~~ $S_n = T_1 + T_2 + T_3 + T_4 + \cdots + T_n$

$$= \frac{3}{4}(\sin\alpha + \sin 2\alpha + \sin 3\alpha + \cdots + \sin n\alpha) - \frac{1}{4}(\sin 3\alpha + \sin 6\alpha + \sin 9\alpha + \cdots + \sin 3n\alpha)$$

$$= \frac{3}{4}\left\{\frac{\sin\left[\alpha + \left(\frac{n-1}{2}\right)\alpha\right]\sin\left(\frac{n\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)}\right\} - \frac{1}{4}\left\{\frac{\sin\left[3\alpha + \left(\frac{n-1}{2}\right)3\alpha\right]\sin\left(\frac{n3\alpha}{2}\right)}{\sin\left(\frac{3\alpha}{2}\right)}\right\}$$

$$= \frac{3}{4}\left\{\frac{\sin\left(\frac{2\alpha + n\alpha - \alpha}{2}\right)\sin\left(\frac{n\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)}\right\} - \frac{1}{4}\left\{\frac{\sin\left[\frac{6\alpha + 3n\alpha - 3\alpha}{2}\right]\sin\left(\frac{3n\alpha}{2}\right)}{\sin\left(\frac{3\alpha}{2}\right)}\right\}$$

$$S_n = \frac{3}{4}\left\{\frac{\sin\left(\frac{n+1}{2}\right)\alpha \ \sin\left(\frac{n\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)}\right\} - \frac{1}{4}\left\{\frac{\sin 3\left(\frac{n+1}{2}\right)\alpha \ \sin\left(\frac{3n\alpha}{2}\right)}{\sin\left(\frac{3\alpha}{2}\right)}\right\}$$

Example: Sum up the series

$$\tan^{-1}\frac{1}{1+1+1^2}+\tan^{-1}\frac{1}{1+2+2^2}+\tan^{-1}\frac{1}{1+3+3^2}+\cdots \text{ n terms}$$

Solution

Let $T_1=\tan^{-1}\frac{1}{1+1+1^2}=\tan^{-1}\frac{1}{3}=\tan^{-1}\frac{2-1}{1+2\cdot 1}=\tan^{-1}2-\tan^{-1}1$

$T_2=\tan^{-1}\frac{1}{1+2+2^2}=\tan^{-1}\frac{1}{7}=\tan^{-1}\frac{3-2}{1+3\cdot 2}=\tan^{-1}3-\tan^{-1}2$

$T_3=\tan^{-1}\frac{1}{1+3+3^2}=\tan^{-1}\frac{1}{13}=\tan^{-1}\frac{4-3}{1+4\cdot 3}=\tan^{-1}4-\tan^{-1}3$

$\cdots \quad \cdots \quad \cdots$

$T_n=\tan^{-1}\frac{1}{1+n+n^2}=\tan^{-1}\frac{(n+1)-n}{1+(n+1)n}=\tan^{-1}(n+1)-\tan^{-1}n$

Adding all the terms, we get

$=T_1+T_2+T_3+\cdots+T_n$

$S_n=-\tan^{-1}1+\tan^{-1}(n+1)$

$=\tan^{-1}(n+1)-\tan^{-1}1$

$=\tan^{-1}\left[\frac{n+1-1}{1+(n+1)(1)}\right]$

$$\boxed{S_n=\tan^{-1}\left(\frac{n}{n+2}\right)}$$

$\cos\theta + i\sin\theta$ Method of Summation

This method can be used to to finding the sum of the series of the form $a_0 \cos\alpha + a_1 \cos(\alpha+\beta) + a_2 \cos(\alpha+2\beta) + \dots$ and

$a_0 \sin\alpha + a_1 \sin(\alpha+\beta) + a_2 \sin(\alpha+2\beta) + \dots$

only when the sum of the series is of the form of G.P.

ie, $a_0 + a_1 x + a_2 x^2 + \dots$

Example Sum of the series

$$1 + \frac{\cos\alpha}{1.\cos\alpha} + \frac{\cos 2\alpha}{2!\cos^2\alpha} + \frac{\cos 3\alpha}{3!\cos^3\alpha} + \dots \infty$$

Solution

Let $C = 1 + \frac{\cos\alpha}{\cos\alpha} + \frac{\cos 2\alpha}{2!\cos^2\alpha} + \frac{\cos 3\alpha}{3!\cos^3\alpha} + \dots$

$S = 0 + \frac{\sin\alpha}{\cos\alpha} + \frac{\sin 2\alpha}{2!\cos^2\alpha} + \frac{\sin 3\alpha}{3!\cos^3\alpha} + \dots$

$$C + iS = 1 + \frac{1}{\cos\alpha}(\cos\alpha + i\sin\alpha) + \frac{1}{2!\cos^2\alpha}(\cos 2\alpha + i\sin 2\alpha)$$
$$+ \frac{1}{3!\cos^3\alpha}(\cos 3\alpha + i\sin 3\alpha) + \dots$$

$$= 1 + \frac{1}{\cos\alpha} e^{i\alpha} + \frac{1}{2!\cos^2\alpha} e^{2i\alpha} + \frac{1}{3!\cos^3\alpha} e^{3i\alpha} + \dots$$

$$= 1 + \sec\alpha\, e^{i\alpha} + \frac{1}{2!}\sec^2\alpha\, e^{2i\alpha} + \frac{1}{3!}\sec^3\alpha\, e^{3i\alpha} + \dots$$

$$= \cancel{1+\sec\alpha}\; 1 + e^{i\alpha}\sec\alpha + \frac{1}{2!}(e^{i\alpha}\sec\alpha)^2 + \frac{1}{3!}(e^{i\alpha}\sec\alpha)^3 + \dots$$

Take $e^{i\alpha}\sec\alpha = z$

$$= 1 + \frac{z}{1!} + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \dots$$

$$= e^z \qquad \left\{\because e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right.$$

$$= (e)^{e^{i\alpha}\cdot \sec\alpha}$$

$$= e^{(\cos\alpha + i\sin\alpha)\sec\alpha}$$

$$= e^{(\cos\alpha + i\sin\alpha)\frac{1}{\cos\alpha}}$$

$$= e^{(1+i\tan\alpha)}$$

$$= e^{1}\cdot e^{i\tan\alpha} \quad \left\{\because e^{i\theta} = \cos\theta + i\sin\theta\right.$$

$$C + iS = e\left[\cos(\tan\alpha) + i\sin(\tan\alpha)\right]$$

Now equating the real and imaginary Parts, we get

$$\boxed{C = e\cdot\cos(\tan\alpha)}$$

Example: Sum up the Series,

$$\cos\alpha + \frac{\cos\alpha}{1!}\cos 2\alpha + \frac{\cos^2\alpha}{2!}\cos 3\alpha + \frac{\cos^3\alpha}{3!}\cos 4\alpha + \dots$$

Solution:

Let us assume that

$$C = \cos\alpha + \frac{\cos\alpha}{1!}\cos 2\alpha + \frac{\cos^2\alpha}{2!}\cos 3\alpha + \frac{\cos^3\alpha}{3!}\cos 4\alpha + \dots$$

and
$$S = \sin\alpha + \frac{\cos\alpha}{1!}\sin 2\alpha + \frac{\cos^2\alpha}{2!}\sin 3\alpha + \frac{\cos^3\alpha}{3!}\sin 4\alpha + \dots$$

Now

$$C + iS = (\cos\alpha + i\sin\alpha) + \frac{\cos\alpha}{1!}(\cos 2\alpha + i\sin 2\alpha) + \frac{\cos^2\alpha}{2!}(\cos 3\alpha + i\sin 3\alpha) + \dots$$

$$= e^{i\alpha} + \frac{\cos\alpha}{1!}e^{2i\alpha} + \frac{\cos^2\alpha}{2!}e^{3i\alpha} + \dots$$

Taking $e^{i\alpha}$ as a Common factor

$$= e^{i\alpha}\left(1 + \frac{\cos\alpha}{1!}e^{i\alpha} + \frac{\cos^2\alpha}{2!}e^{2i\alpha} + \dots\right)$$

$$= e^{i\alpha}\left\{1 + \frac{\cos\alpha\cdot e^{i\alpha}}{1!} + \frac{(\cos\alpha\cdot e^{i\alpha})^2}{2!} + \frac{(\cos\alpha\cdot e^{i\alpha})^3}{3!} + \dots\right\}$$

Take $z = \cos\alpha \cdot e^{i\alpha}$

Then,

$$C + iS = e^{i\alpha}\left\{1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots\right\}$$

$$= e^{i\alpha} \cdot e^{z}$$

$$= e^{i\alpha}\, e^{\cos\alpha \cdot e^{i\alpha}}$$

$$= e^{i\alpha} \cdot e^{\cos\alpha(\cos\alpha + i\sin\alpha)}$$

$$= e^{i\alpha} \cdot e^{\cos^2\alpha + i\sin\alpha\cdot\cos\alpha}$$

$$= e^{i\alpha}\, e^{\cos^2\alpha} \cdot e^{i\sin\alpha\cos\alpha}$$

$$= e^{\cos^2\alpha} \cdot e^{i(\alpha + \sin\alpha\cos\alpha)}$$

$\{\because e^{i\theta} = \cos\theta + i\sin\theta$

$$C + iS = e^{\cos^2\alpha}\left[\cos(\alpha + \sin\alpha\cos\alpha) + i\sin(\alpha + \sin\alpha\cos\alpha)\right]$$

Equating the real and imaginary parts, we get

$$C = e^{\cos^2\alpha}\cos(\alpha + \sin\alpha\cos\alpha)$$

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Example Sum to infinity $\sin\alpha + x\sin(\alpha+\beta) + \frac{x^2}{2!}\sin(\alpha+2\beta) + \cdots$

Solution From the given series, let us assume that

$$S = \sin\alpha + \frac{x}{1!}\sin(\alpha+\beta) + \frac{x^2}{2!}\sin(\alpha+2\beta) + \cdots$$

~~and~~ $$C = \cos\alpha + \frac{x}{1!}\cos(\alpha+\beta) + \frac{x^2}{2!}\cos(\alpha+2\beta) + \cdots$$

$$C+iS = (\cos\alpha + i\sin\alpha) + \frac{x}{1!}[\cos(\alpha+\beta) + i\sin(\alpha+\beta)] + \frac{x^2}{2!}[\cos(\alpha+2\beta) + i\sin(\alpha+2\beta)] + \cdots$$

$$= e^{i\alpha} + \frac{x}{1!}e^{i(\alpha+\beta)} + \frac{x^2}{2!}e^{i(\alpha+2\beta)} + \cdots$$

$$= e^{i\alpha} + \frac{x}{1!}e^{i\alpha}e^{i\beta} + \frac{x^2}{2!}e^{i\alpha}\cdot e^{i2\beta} + \cdots$$

Taking $e^{i\alpha}$ as Common

$$= e^{i\alpha}\left\{1 + \frac{x}{1!}e^{i\beta} + \frac{x^2}{2!}e^{i2\beta} + \cdots\right\}$$

$$= e^{i\alpha}\left\{1 + \left(\frac{x}{1!}e^{i\beta}\right) + \frac{(x\,e^{i\beta})^2}{2!} + \cdots\right\}$$

$$= e^{i\alpha}\cdot e^{\frac{x}{1!}e^{i\beta}}$$

$$= e^{i\alpha}\cdot e^{x(\cos\beta + i\sin\beta)}$$

$$= e^{i\alpha}\cdot e^{x\cos\beta}\cdot e^{ix\sin\beta}$$

$$= e^{x\cos\beta}\cdot e^{i(\alpha + x\sin\beta)}$$

$$C+iS = e^{x\cos\beta}\{\cos(\alpha + x\sin\beta) + i\sin(\alpha + x\sin\beta)\}$$

Equating the imaginary Part, we get

$$S = e^{x\cos\beta}[\sin(\alpha + x\sin\beta)]$$

Note: $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$

Gregory's Series

If $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$, θ can be expanded as a power series ~~of~~ ~~θ~~ interms of θ and then the series is called Gregory's series

Let $e^{i\theta} = \cos\theta + i\sin\theta$

$$= \cos\theta\left(1 + \frac{i\sin\theta}{\cos\theta}\right)$$

$$e^{i\theta} = \cos\theta\,(1 + i\tan\theta)$$

Taking log on both sides

$$\log_e e^{i\theta} = \log \cos\theta\,(1+i\tan\theta) \qquad \{\because \log MN = \log M + \log N$$

$$\Rightarrow \quad i\theta = \log\cos\theta + \log(1 + i\tan\theta)$$

$$= \log\cos\theta + i\tan\theta - \frac{(i\tan\theta)^2}{2} + \frac{(i\tan\theta)^3}{3} - \dots$$

Combining the real and imaginary parts on RHS

$$i\theta = \left(\log\cos\theta + \frac{\tan^2\theta}{2} - \frac{\tan^4\theta}{4} + \dots\infty\right)$$

$$+ i\left(\tan\theta - \frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5} - \dots\infty\right)$$

Now equating the imaginary part,

$$\theta = \tan\theta - \frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5} - \dots\infty \rightarrow (1)$$

This series is called Gregory's Series.

Note: (1) Another form of gregory's series is, ~~tan~~ obtained by putting $\theta = \tan^{-1}x$

then (1) $\Rightarrow$ $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots\infty$; $(-1 \le x \le 1)$

(2) Gregory's Series can be used to ~~fi~~ evaluate the value of π.

Example using Gregory's Series, Show that,

$$\frac{\pi}{2\sqrt{3}} = 1 - \frac{1}{3\cdot 3} + \frac{1}{5\cdot 3^2} - \frac{1}{7\cdot 3^3} + \cdots \infty$$

Solution

$$\boxed{x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \infty = \tan^{-1}x \quad \text{is Called Gregory's Series } (-1 \le x \le 1)}$$

$$RHS = 1 - \frac{1}{3\cdot 3} + \frac{1}{5\cdot 3^2} - \frac{1}{7\cdot 3^3} + \cdots \infty$$

It can be rewritten as

$$= \sqrt{3}\left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}\cdot 3\cdot 3} + \frac{1}{\sqrt{3}\cdot 5\cdot 3^2} - \cdots \infty\right)$$

$$= \sqrt{3}\left\{\left(\frac{1}{\sqrt{3}}\right) - \frac{1}{3(\sqrt{3})^3} + \frac{1}{5(\sqrt{3})^5} - \cdots \infty\right)$$

Take $\frac{1}{\sqrt{3}} = x$

$$= \sqrt{3}\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots \infty\right)$$

$$= \sqrt{3}\,(\tan^{-1}x) \qquad \{\because \text{By Gregory's series}$$

$$= \sqrt{3}\left(\tan^{-1}\frac{1}{\sqrt{3}}\right)$$

$$= \sqrt{3}\left(\frac{\pi}{6}\right)$$

$$= \sqrt{3}\left(\frac{\pi}{2\times 3}\right)$$

$$= \frac{\pi}{2\sqrt{3}}$$

$$= LHS$$

Example Prove that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$

Proof: Gregory's Series

$$\boxed{\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots \infty} \qquad -1 \le x \le 1$$

$$LHS = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

$$= 1 - \frac{1^3}{3} + \frac{1^5}{5} - \frac{1^7}{7} + \dots$$

$$= \tan^{-1}(1)$$

$$= \frac{\pi}{4}$$

$$= RHS$$

Example Prove that $\frac{1}{2} - \frac{1}{3}\frac{1}{2^3} + \frac{1}{5}\frac{1}{2^5} - \frac{1}{7}\frac{1}{2^7} + \dots \infty = \tan^{-1}\left(\frac{1}{2}\right)$

$$LHS = \left(\frac{1}{2}\right) - \frac{(1/2)^3}{3} + \frac{(1/2)^5}{5} - \frac{(1/2)^7}{7} + \dots \infty$$

By Gregory's Series

$$= \tan^{-1}\left(\frac{1}{2}\right)$$

$$= RHS$$

Example $\frac{1}{3} - \frac{1}{3} \cdot \frac{1}{3^3} + \frac{1}{5} \cdot \frac{1}{3^5} - \frac{1}{7} \cdot \frac{1}{3^7} + \dots \infty = \tan^{-1}\left(\frac{1}{3}\right)$

$$LHS = \left(\frac{1}{3}\right) - \frac{(1/3)^3}{3} + \frac{(1/3)^5}{5} - \frac{(1/3)^7}{7} + \dots \infty$$

$$= \tan^{-1}(1/3)$$

$$= RHS$$

Euler's Series

The Series for,

$$\frac{\pi}{4} = \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right)$$ is Called Euler's series.

$\cancel{\frac{\pi}{4} = \tan^{-1}\left(\frac{1}{2}\right.}$ By Gregory's Series,

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \infty$$

$$\therefore \tan^{-1}\frac{1}{2} = \frac{1}{2} - \frac{1}{3}\left(\frac{1}{2}\right)^3 + \frac{1}{5}\left(\frac{1}{2}\right)^5 - \cdots$$

$$\boxed{\tan^{-1}\left(\frac{1}{2}\right) = \frac{1}{2} - \frac{1}{3}\left(\frac{1}{2^3}\right) + \frac{1}{5}\left(\frac{1}{2^5}\right) - \cdots} \longrightarrow (1)$$

$$\boxed{\tan^{-1}\left(\frac{1}{3}\right) = \frac{1}{3} - \frac{1}{3}\left(\frac{1}{3^3}\right) + \frac{1}{5}\left(\frac{1}{3^5}\right) - \cdots} \longrightarrow (2)$$

Adding (1) & (2)

$$\boxed{\frac{\pi}{4} = \left(\frac{1}{2} + \frac{1}{3}\right) - \frac{1}{3}\left(\frac{1}{2^3} + \frac{1}{3^3}\right) + \frac{1}{5}\left(\frac{1}{2^5} + \frac{1}{3^5}\right) + \cdots}$$